# Information transmission in quantum measurement processes with pure operation



1999 J. Phys. A: Math. Gen. 32 6527

(http://iopscience.iop.org/0305-4470/32/37/304)

J. Phys. A: Math. Gen. **32** (1999) 6527–6533. Printed in the UK PII: S0305-4470(99)04564-3

# Information transmission in quantum measurement processes with pure operation

Masashi Ban

Advanced Research Laboratory, Hitachi Ltd, Akanuma 2520, Hatoyama, Saitama 350-0395, Japan

Received 25 May 1999

**Abstract.** Quantum and classical informations are considered in quantum measurement processes described by pure operations. The quantum information is given by coherent information in the state change of the measured physical system, and the classical information represented by the Shannon mutual information is obtained from the measurement outcomes. It is shown that if the maximum classical information is obtained, the quantum information becomes zero and if any classical information is not obtained, all the quantum information that the physical system has can be transmitted.

When we perform a quantum measurement of a physical system to obtain some information on a physical quantity, the quantum state of the measured system inevitably changes due to the effects of the quantum measurement. Such a state change is equivalent to a noisy quantum channel, both of which are described by a completely positive map [1–3]. Thus we can consider how much quantum information is transmitted through this noisy quantum channel induced by the quantum measurement process [4–8]. On the other hand, we can obtain the classical information (Shannon information) on the physical quantity from the measurement outcomes. This means that there is a communication channel from the physical system to the measurement apparatus. Therefore a quantum measurement process has two communication channels; one transmits quantum information of a physical system to be measured and the other the classical information on a physical quantity. The purpose of this paper is to investigate the information transmission in quantum measurement processes described by pure quantum operations. In particular, it is shown that if we obtain the maximum classical information from the measurement outcome, the quantum information cannot be transmitted through the noisy quantum channel and if we cannot obtain any classical information, all the quantum information that the physical system has can be transmitted.

Suppose that we measure some intrinsic observable $\hat{X}^S$, having a discrete spectrum, of a physical system $\mathcal{S}$, where we denote the projection-valued measure of this observable, corresponding to the eigenvalue $x$, as $\hat{\mathcal{X}}^S(x) = |\psi^S(x)\rangle\langle\psi^S(x)|$ and the spectral set as $\Omega_X$. Thus the spectral decomposition is given by $\hat{X}^S = \sum_{x\in\Omega_X} x\hat{\mathcal{X}}^S(x)$. To perform the quantum measurement, we first prepare an appropriate measurement apparatus $\mathcal{A}$ and then we make an interaction between the physical system and the measurement apparatus to create some quantum correlation between them, where the unitary operator which describes the evolution of the system–apparatus compound system is denoted as $\hat{U}^{SA}$. The readout process of the result $y$ shown by the measurement apparatus is described by a positive operator-valued measure

  

$\hat{\mathcal{Y}}^A(y)$, satisfying the relations $\hat{\mathcal{Y}}^A(y) \geqslant 0$ and $\sum_{y\in\Omega_Y} \hat{\mathcal{Y}}^A(y) = \hat{1}^A$, where $\Omega_Y$ is the set of all possible measurement outcomes. If the quantum state of the physical system before the interaction with the measurement apparatus is described by a statistical operator $\hat{\rho}_{\text{in}}^S$ and the measurement apparatus is prepared in a quantum state $\rho_{\text{in}}^A$, the quantum state of the system–apparatus compound system just before the readout of the measurement outcome is given by $\hat{\rho}_{\text{out}}^{SA} = \hat{U}^{SA}(\hat{\rho}_{\text{in}}^S \otimes \hat{\rho}_{\text{in}}^A)\hat{U}^{SA\dagger}$. Then the probability $P_{\text{out}}^A(y)$ of the measurement outcome $y$ and the quantum state $\hat{\rho}_{\text{out}}^S(y)$ of the physical system after obtaining the measurement outcome $y$ are given by the following formulae [1–3]:

$$P_{\text{out}}^A(y) = \text{Tr}_S[\hat{\mathcal{L}}_y^S \hat{\rho}_{\text{in}}^S] \qquad \hat{\rho}_{\text{out}}^S(y) = \frac{\hat{\mathcal{L}}_y^S \hat{\rho}_{\text{in}}^S}{\text{Tr}_S[\hat{\mathcal{L}}_y^S \hat{\rho}_{\text{in}}^S]} \tag{1}$$

where $\hat{\mathcal{L}}_y^S$ is the quantum operation of the physical system, which is defined for an arbitrary operator $\hat{O}^S$ of the physical system

$$\hat{\mathcal{L}}_y^S \hat{O}^S = \text{Tr}_A[(\hat{1}^S \otimes \hat{\mathcal{Y}}^A(y))\hat{U}^{SA}(\hat{O}^S \otimes \hat{\rho}_{\text{in}}^A)\hat{U}^{SA\dagger}]. \tag{2}$$

Since we can always exclude the values $y$ that cannot be obtained by the quantum measurement, we assume that $P_{\text{out}}^A(y) \neq 0$ for all $y \in \Omega_Y$ in equation (1), without loss of generality. The quantum operation $\hat{\mathcal{L}}_y^S$ is a trace-decreasing completely positive map and thus it can be represented in the following form [2]

$$\hat{\mathcal{L}}_y^S \hat{O}^S = \sum_\mu \hat{A}_\mu^S(y)\hat{O}^S \hat{A}_\mu^{S\dagger}(y) \qquad \sum_\mu \hat{A}_\mu^{S\dagger}(y)\hat{A}_\mu^S(y) \leqslant \hat{1}^S \tag{3}$$

where $\hat{A}_\mu^S(y)$ is an operator determined by $\hat{U}^{SA}$, $\hat{\mathcal{Y}}^A(y)$ and $\hat{\rho}_{\text{in}}^A$. This equation is referred to as the operator-sum representation [4].

When the operator-sum representation equation (3) has only one operator $\hat{A}^S(y)$, the quantum operation $\hat{\mathcal{L}}_y^S(y)$ is called an ideal quantum operation [9] or a pure quantum operation [10]. In quantum measurement processes described by pure quantum operations, the quantum state $\hat{\rho}_{\text{out}}^S(y)$ becomes pure if the initial quantum state $\hat{\rho}_{\text{in}}^S$ is pure. Pure quantum operations appear in many quantum measurement processes. In fact, when a measurement apparatus is prepared in a pure quantum state $\hat{\rho}_{\text{in}}^A = |\phi_{\text{in}}^A\rangle\langle\phi_{\text{in}}^A|$ and the measurement outcomes $y$ is provided by measuring the pointer observable of the measurement apparatus, we obtain the pure quantum operation

$$\hat{\mathcal{L}}_y^S \hat{O}^S = \hat{A}^S(y)\hat{O}^S \hat{A}^{S\dagger}(y) \qquad \hat{A}^S(y) = \langle\phi^A(y)|\hat{U}^{SA}|\phi_{\text{in}}^A\rangle \tag{4}$$

where $|\phi^A(y)\rangle$ is the eigenstate of the pointer observable with the eigenvalue $y$. In the rest of this paper, we confine ourselves to considering the quantum measurement process described by the pure quantum operation.

In previous papers [11–13], we have investigated the information-theoretical properties of quantum measurement processes and we have obtained the necessary and sufficient condition under that the amount of information on the intrinsic observable $\hat{X}^S$, obtained from the measurement outcome, can be represented by the Shannon mutual information. The condition is that the intrinsic observable $\hat{\mathcal{X}}^S(x)$ [or $\hat{X}^S$] of the physical system commutes with the operational observable given by $\hat{\mathcal{X}}_{\text{op}}^S(y) = \hat{\mathcal{L}}_y^{S\dagger}\hat{1}^S$; that is,

$$[\hat{\mathcal{X}}^S(x), \hat{\mathcal{X}}_{\text{op}}^S(y)] = 0 \qquad (\forall\, x \in \Omega_X \quad \forall\, y \in \Omega_Y). \tag{5}$$

When the quantum measurement process is described by the pure operation, the operational observable becomes $\hat{\mathcal{X}}_{\text{op}}^S(y) = \hat{A}^{S\dagger}(y)\hat{A}^S(y)$. If the relation given by equation (5) holds, the amount of information on the intrinsic observable of the physical system obtained by the quantum measurement is given by

$$I(X_{\rm in}^S : Y_{\rm out}^A) = \sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P_{\rm c}^A(y|x)P_{\rm in}^S(x)\log\left[\frac{P_{\rm c}^A(y|x)}{P_{\rm out}^A(y)}\right] \tag{6}$$

where $P_{\rm in}^S(x) = \langle\psi^S(x)|\hat{\rho}_{\rm in}^S|\psi^S(x)\rangle$ is the probability that the intrinsic observable takes a value $x$ in the quantum state $\hat{\rho}_{\rm in}^S$ and the conditional probability $P_{\rm c}^A(y|x)$ that the measurement outcome $y$ is obtained when the intrinsic observable takes a value $x$ in the quantum state $\hat{\rho}_{\rm in}^S$ is given by

$$P_{\rm c}^A(y|x) = \langle\psi^S(x)|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(x)\rangle \tag{7}$$

which satisfies the relation

$$P_{\rm out}^A(y) = \sum_{x\in\Omega_X} P_{\rm c}^A(y|x)P_{\rm in}^S(x). \tag{8}$$

Furthermore, we have the relation between the intrinsic and operational observables

$$\hat{A}^{S\dagger}(y)\hat{A}^S(y) = \sum_{x\in\Omega_X} P_{\rm c}^A(y|x)\hat{\mathcal{X}}^S(x). \tag{9}$$

The Shannon mutual information $I(X_{\rm in}^S : Y_{\rm out}^A)$ given by equation (6) represents how much classical information on the intrinsic observable $\hat{\mathcal{X}}^S(x)$ is transmitted from the physical system before performing the quantum measurement to the observer who performed the quantum measurement. It is a wellknown fact that the mutual information satisfies the inequality $0 \leqslant I(X_{\rm in}^S : Y_{\rm out}^A) \leqslant H(X_{\rm in}^S)$, where $H(X_{\rm in}^S)$ is the Shannon entropy of the intrinsic observable in the quantum state $\hat{\rho}_{\rm in}^S$ of the physical system, that is, $H(X_{\rm in}^S) = \sum_{x\in\Omega_X} P_{\rm in}^S(x)\log P_{\rm in}^S(x)$.

When we perform the quantum measurement of the physical system and we obtain the measurement outcome $y$, the quantum state of the measured physical system changes as $\hat{\rho}_{\rm in}^S \to \hat{\rho}_{\rm out}^S(y)$. Such a state change is equivalent to a noisy quantum channel described by the trace-decreasing completely positive map $\hat{\mathcal{L}}_y^S$. Thus we can consider the transmission of the quantum information from the physical system before performing the quantum measurement to the physical system after obtaining the measurement outcome. The amount of quantum information transmitted through the noisy quantum channel is quantified by the coherent information [5–7]. The coherent information $I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S)$ of the noisy quantum channel described by $\hat{\mathcal{L}}_y^S\hat{O}^S = \sum_\mu \hat{A}_\mu^S(y)\hat{Q}^S\hat{A}_\mu^{S\dagger}(y)$ is given by

$$I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S) = S\left(\frac{\hat{\mathcal{L}}_y^S\hat{\rho}_{\rm in}^S}{{\rm Tr}_S[\hat{\mathcal{L}}_y^S\hat{\rho}_{\rm in}^S]}\right) - S_{\rm e}(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S) \tag{10}$$

where $S(\hat{\rho}) = -{\rm Tr}[\hat{\rho}\log\hat{\rho}]$ is the von Neumann entropy and $S_{\rm e}(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S)$ is the entropy exchange of the noisy quantum channel [4, 7]

$$S_{\rm e}(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S) = -{\rm Tr}[W(y)\log W(y)] \qquad W_{\mu\nu}(y) = \frac{{\rm Tr}_S[\hat{A}_\mu^S(y)\hat{\rho}_{\rm in}^S\hat{A}_\nu^{S\dagger}(y)]}{{\rm Tr}_S[\hat{\mathcal{L}}_y^S\hat{\rho}_{\rm in}^S]}. \tag{11}$$

It should be noted that the coherent information can take negative values. It is easy to see from equation (11) that the entropy exchange $S_{\rm e}(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S)$ vanishes in the quantum measurement process described by the pure operation and the coherent information $I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S)$ becomes

$$I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S) = S\left(\frac{\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)}{{\rm Tr}_S[\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)]}\right) \tag{12}$$

which is non-negative. The noisy quantum channel $\hat{\mathcal{L}}_y^S$ is obtained with probability $P_{\rm out}^A(y)$. Then it can be proved that the average value of the coherent information satisfies the inequality

$$\langle I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S)\rangle_y = \sum_{y\in\Omega_Y} P_{\rm out}^A(y) I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_y^S) \leqslant S(\hat{\rho}_{\rm in}^S). \tag{13}$$

The coherent information in the quantum information theory plays the same role as the Shannon mutual information in the classical information theory does [5–7].

We now consider the relation between the classical information $I(X_{\rm in}^S : Y_{\rm out}^A)$ on the intrinsic observable and the quantum information $I_C(\hat{\rho}_{\rm in}^S, \hat{\mathcal{L}}_y^S)$ transmitted through the noisy quantum channel. We first suppose that we obtain the maximum information on the intrinsic observable of the physical system: namely, $I(X_{\rm in}^S : Y_{\rm out}^A) = H(X_{\rm in}^S)$ and we cannot obtain any further information even if we repeat the same quantum measurement since we have already obtained the maximum information. In this case, the conditional probability becomes $P_{\rm c}(y|x) = \delta_{y,f(x)}$ and the operator $\hat{A}^S(y)$ is given by $\hat{A}^S(y) = |\psi^S(\tilde{x})\rangle\langle\psi^S(\tilde{x})|$, where the function $f(x)$ is invertible and $\tilde{x} = f^{-1}(y)$. This result indicates that the quantum state of the physical system after obtaining the measurement outcome $y$ becomes a pure state

$$\hat{\rho}_{\rm out}^S(y) = \frac{\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)}{{\rm Tr}_S[\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)]} = |\psi^S(\tilde{x})\rangle\langle\psi^S(\tilde{x})|. \tag{14}$$

Thus, it is found from equation (12) that when we obtain the maximum information on the intrinsic observable of the physical system the coherent information becomes zero: that is, $I_C(\hat{\rho}_{\rm in}^S, \hat{\mathcal{L}}_y^S) = 0$.

We next consider the case that we cannot obtain any information on the intrinsic observable of the physical system: that is, $I(X_{\rm in}^S : Y_{\rm out}^A) = 0$. In this case, since the conditional probability $P_{\rm c}^A(y|x)$ does not depend on $x$, we obtain the following relation from equation (7) or (9):

$$\langle\psi^S(x)|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(x)\rangle = \langle\psi^S(x')|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(x')\rangle. \tag{15}$$

Furthermore, we obtain from equation (5) or (9)

$$\langle\psi^S(x)|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(x')\rangle = 0 \qquad (x \neq x'). \tag{16}$$

Using equations (15) and (16), we can calculate the von Neumann entropy $S(\hat{\rho}_{\rm out}^S(y))$. We first expand the statistical operator $\hat{\rho}_{\rm in}^S$ as

$$\hat{\rho}_{\rm in}^S = \sum_{x\in\Omega_X}\sum_{x'\in\Omega_X} f(x,x')|\psi^S(x)\rangle\langle\psi^S(x')| \qquad \sum_{x\in\Omega_X} f(x,x) = 1. \tag{17}$$

Since we obtain from equations (15)–(17)

$${\rm Tr}_S[\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)] = \langle\psi^S(x)|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(x)\rangle \equiv F(y) \tag{18}$$

the quantum state $\hat{\rho}_{\rm out}^S(y)$ of the physical system after obtaining the measurement outcome $y$ becomes

$$\hat{\rho}_{\rm out}^S(y) = \frac{\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)}{{\rm Tr}_S[\hat{A}^S(y)\hat{\rho}_{\rm in}^S\hat{A}^{S\dagger}(y)]} = \sum_{x\in\Omega_X}\sum_{x'\in\Omega_X} f(x,x')|\tilde{\psi}^S(x)\rangle\langle\tilde{\psi}^S(x')| \tag{19}$$

with

$$|\tilde{\psi}^S(x)\rangle = \frac{\hat{A}^S(y)|\psi^S(x)\rangle}{\sqrt{F(y)}} \qquad \langle\tilde{\psi}^S(x)|\tilde{\psi}^S(x')\rangle = \delta_{x,x'}. \tag{20}$$

It is found from equations (17) and (19) that the eigenvalues of the statistical operator $\hat{\rho}_{\rm out}^S(y)$ are equal to those of the statistical operator $\hat{\rho}_{\rm in}^S$ and thus the equality $S(\hat{\rho}_{\rm out}^S(y)) = S(\hat{\rho}_{\rm in}^S)$ holds. Therefore, when we cannot obtain any information on the intrinsic observable of the physical system, the coherent information is equal to the von Neumann entropy of the physical system in the quantum state $\hat{\rho}^S_{\rm in}$: that is, $I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y) = S(\hat{\rho}^S_{\rm in})$.

We now suppose that the coherent information $I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y)$ vanishes in the state change caused by the quantum measurement. Since the quantum measurement is described by the pure operation, the equality $I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y) = 0$ means that the quantum state $\hat{\rho}^S_{\rm out}(y)$ is pure, namely

$$\left(\frac{\hat{A}^S(y)\hat{\rho}^S_{\rm in}\hat{A}^{S\dagger}(y)}{{\rm Tr}_S[\hat{A}^S(y)\hat{\rho}^S_{\rm in}\hat{A}^{S\dagger}(y)]}\right)^2 = \frac{\hat{A}^S(y)\hat{\rho}^S_{\rm in}\hat{A}^{S\dagger}(y)}{{\rm Tr}_S[\hat{A}^S(y)\hat{\rho}^S_{\rm in}\hat{A}^{S\dagger}(y)]}. \tag{21}$$

Using the relation $\hat{A}^{S\dagger}(y)\hat{A}^S(y) = \sum_{x\in\Omega_X} P^A_{\rm c}(y|x)\hat{\mathcal{X}}^S(x)$, we obtain from this equation

$$\sum_{x\in\Omega_X}\sum_{x'\in\Omega_X} P^A_{\rm c}(y|x)P^A_{\rm c}(y|x')|\langle\psi^S(x)|\hat{\rho}^S_{\rm in}|\psi^S(x')\rangle|^2 = \left(\sum_{x\in\Omega_X} P^A_{\rm c}(y|x)\langle\psi^S(x)|\hat{\rho}^S_{\rm in}|\psi^S(x)\rangle\right)^2. \tag{22}$$

Since the quantum state $\hat{\rho}^S_{\rm in}$ of the physical system is arbitrary, equation (22) should be satisfied even if the statistical operator $\hat{\rho}^S_{\rm in}$ is diagonal with respect to the eigenstates of the intrinsic observable. Thus we obtain

$$\sum\sum_{\substack{x,x'\in\Omega_X\\(x\neq x')}} P^A_{\rm c}(y|x)P^A_{\rm c}(y|x')\langle\psi^S(x)|\hat{\rho}^S_{\rm in}|\psi^S(x)\rangle\langle\psi^S(x')|\hat{\rho}^S_{\rm in}|\psi^S(x')\rangle = 0 \tag{23}$$

which means that $P^A_{\rm c}(y|x)P^A_{\rm c}(y|x') = 0$ $(x \neq x')$. Therefore, the conditional probability can be expressed as $P^A_{\rm c}(y|x) = \delta_{y,f(x)}$, where $f(x) \neq f(x')$ for $x \neq x'$. We see from this result that the information on the intrinsic observable, obtained by the quantum measurement, becomes maximum and the equality $I(X^S_{\rm in} : Y^A_{\rm out}) = H(X^S_{\rm in})$ holds.

We next suppose that the coherent information is equal to the von Neumann entropy of the physical system in the quantum state $\hat{\rho}^S_{\rm in}$: that is, $I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y) = S(\hat{\rho}^S_{\rm in})$. In this case, the quantum channel $\hat{\mathcal{L}}^S_y$ becomes reversible, which means that there is a trace-preserving completely positive map $\hat{\mathcal{R}}^S$ such that $\hat{\mathcal{R}}^S(\hat{\mathcal{L}}^S_y\hat{\rho}^S_{\rm in}/\,{\rm Tr}_S[\hat{\mathcal{L}}^S_y\hat{\rho}^S_{\rm in}]) = \hat{\rho}^S_{\rm in}$ [5, 10]. The condition of this reversibility is equivalent to that given by equations (15) and (16) [10, 14]. Thus, the conditional probability $P^A_{\rm c}(y|x)$ becomes independent of $x$. This result means that we cannot obtain any information on the intrinsic observable of the physical system and the equality $I(X^S_{\rm in} : Y^A_{\rm out}) = 0$ is established.

Therefore, the results that we have obtained are summarized in the following relations:

$$I(X^S_{\rm in} : Y^A_{\rm out}) = H(X^S_{\rm in}) \Longleftrightarrow I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y) = 0 \Longleftrightarrow \langle I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}^S_y)\rangle_y = 0 \tag{24}$$

$$I(X^S_{\rm in} : Y^A_{\rm out}) = 0 \Longleftrightarrow I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_y) = S(\hat{\rho}^S_{\rm in}) \Longleftrightarrow \langle I_C(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}^S_y)\rangle_y = S(\hat{\rho}^S_{\rm in}) \tag{25}$$

where we have used the inequality equation (13) and the fact that the coherent information is non-negative for pure quantum operations. If we obtain the maximum information on the intrinsic observable of the physical system, the quantum information cannot be transmitted from the physical system before performing the quantum measurement to the physical system after obtaining the measurement outcome. On the other hand, if we do not obtain any information on the intrinsic observable, all the quantum information that the physical system has in the quantum state $\hat{\rho}^S_{\rm in}$ can be transmitted. Furthermore, it is interesting to note that the relations given by equations (15) and (16) are equivalent to the necessary and sufficient condition under which the quantum error correction is possible for the noisy quantum channel $\hat{\mathcal{L}}^S_y$ [10,14]. The condition is expressed as

$$\langle\psi^S(x)|\hat{A}^{S\dagger}(y)\hat{A}^S(y)|\psi^S(y)\rangle = \delta_{x,x'}F(y) \tag{26}$$

where $F(y)$ is independent of $x$. Our results indicate that the condition for the diagonality of the operator $\hat{A}^{S\dagger}(y)\hat{A}^{S}(y)$ is equivalent to that for the existence of the conditional probability $P_{\mathrm{c}}^{A}(y|x)$ of the measurement outcome $y$ for given $x$ in the quantum measurement process. The fact that the function $F(y)$ does not depend on $x$ indicates that the conditional probability $P_{\mathrm{c}}^{A}(y|x)$ becomes equal to the output probability $P_{\mathrm{out}}^{A}(y)$ of the measurement outcome $y$ in the quantum measurement process.

We have considered the two extreme cases. One is the quantum measurement in which the information gain attains maximum and the other is the quantum measurement in which any information cannot be obtained. To consider the intermediate case, we assume the simple model of the quantum measurement process. Here we note that the operational and intrinsic observables satisfies equation (9). Then we assume that the operator $\hat{A}^{S}(y)$ is given by

$$\hat{A}^{S}(y) = \sum_{x\in\Omega_X} \sqrt{P_{\mathrm{c}}^{A}(y|x)}\hat{\mathcal{X}}^{S}(x). \tag{27}$$

We first consider the case that the quantum state of the physical system before performing the quantum measurement is the statistical mixture of the eigenstates of the intrinsic observable, namely, $\hat{\rho}_{\mathrm{in}}^{S} = \sum_{x\in\Omega_X} P_{\mathrm{in}}(x)\hat{\mathcal{X}}^{S}(x)$. Thus we obtain the quantum state $\hat{\rho}_{\mathrm{out}}^{S}(y)$ after obtaining the measurement outcome $y$:

$$\hat{\rho}_{\mathrm{out}}^{S}(y) = \frac{\sum_{x\in\Omega_X} P_{\mathrm{c}}^{A}(y|x)P_{\mathrm{in}}^{S}(x)\hat{\mathcal{X}}^{S}(x)}{P_{\mathrm{out}}^{A}(y)}. \tag{28}$$

Then, the average value of the coherent information is calculated to be

$$\begin{aligned}\langle I_C(\hat{\rho}_{\mathrm{in}}, \hat{\mathcal{L}}_y^{S})\rangle_y &= -\sum_{y\in\Omega_Y} P_{\mathrm{out}}^{A}(y)\sum_{x\in\Omega_X}\left[\frac{P_{\mathrm{c}}^{A}(y|x)P_{\mathrm{in}}^{S}(x)}{P_{\mathrm{out}}^{A}(y)}\right]\log\left[\frac{P_{\mathrm{c}}^{A}(y|x)P_{\mathrm{in}}^{S}(x)}{P_{\mathrm{out}}^{A}(y)}\right]\\ &= H(X_{\mathrm{in}}) - I(X_{\mathrm{in}}^{S} : Y_{\mathrm{out}}^{A})\end{aligned} \tag{29}$$

where the equality $S(\hat{\rho}_{\mathrm{in}}^{S}) = H(X_{\mathrm{in}}^{S})$ is satisfied. Therefore, when the initial quantum state of the physical system is the statistical mixture of the eigenstates of the intrinsic observable, the sum of the averaged coherent information and the Shannon mutual information remains constant, the value of which is the Shannon or von Neumann entropy of the physical system before performing the quantum measurement.

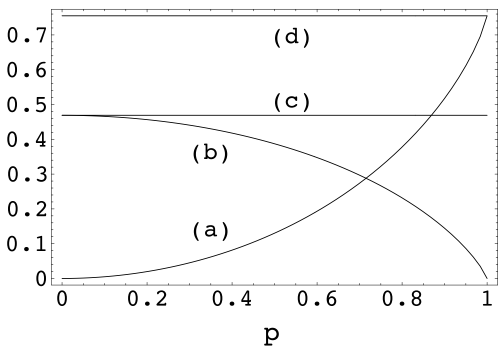


**Figure 1.** Plots of the average value of the coherent information (*a*), the Shannon mutual information (*b*), the von Neumann entropy (*c*) and the Shannon entropy (*d*) before performing the quantum measurement, where we set $r = 0.8$ and $\theta = \pi/4$. In the figure, all the qualities are measured in bits.

We next consider the case that the physical system is a spin-$\frac{1}{2}$ system, where the initial quantum state $\hat{\rho}^S_{\rm in}$ and the intrinsic observable $\hat{\mathcal{X}}^S(x)$ are given by

$$\hat{\rho}^S_{\rm in} = \begin{pmatrix} \frac{1}{2}(1 + r\cos\theta) & \frac{1}{2}r\sin\theta \\ \frac{1}{2}r\sin\theta & \frac{1}{2}(1 - r\cos\theta) \end{pmatrix} \qquad \hat{\mathcal{X}}^S(\uparrow) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{\mathcal{X}}^S(\downarrow) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

with $0 \leqslant r \leqslant 1$. The conditional probability $P^A_{\rm c}(y|x)$ is assumed to be

$$P^A_{\rm c}(\uparrow\,|\,\uparrow) = P^A_{\rm c}(\downarrow\,|\,\downarrow) = \tfrac{1}{2}(1 + p) \qquad P^A_{\rm c}(\uparrow\,|\,\downarrow) = P^A_{\rm c}(\uparrow\,|\,\downarrow) = \tfrac{1}{2}(1 - p) \tag{31}$$

with $0 \leqslant p \leqslant 1$, where we obtain the maximum information if $p = 1$ and we cannot obtain any information if $p = 0$. Then the operator $\hat{A}^S(y)$ becomes

$$\hat{A}^S(\uparrow) = \begin{pmatrix} \sqrt{\frac{1}{2}(1+p)} & 0 \\ 0 & \sqrt{\frac{1}{2}(1-p)} \end{pmatrix} \qquad \hat{A}^S(\downarrow) = \begin{pmatrix} \sqrt{\frac{1}{2}(1-p)} & 0 \\ 0 & \sqrt{\frac{1}{2}(1+p)} \end{pmatrix}. \tag{32}$$

After straightforward calculation, we can obtain

$$I(X^S_{\rm in} : Y^A_{\rm out}) = \mathcal{H}(pr\cos\theta) - \mathcal{H}(p) \qquad H(X^S_{\rm in}) = \mathcal{H}(r\cos\theta) \tag{33}$$

$$I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow}) = \mathcal{H}(\lambda_+) \qquad I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\downarrow}) = \mathcal{H}(\lambda_-) \qquad S(\hat{\rho}^S_{\rm in}) = \mathcal{H}(r) \tag{34}$$

$$\langle I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow\downarrow})\rangle_{\uparrow\downarrow} = \tfrac{1}{2}(1 + pr\cos\theta)\mathcal{H}(\lambda_+) + \tfrac{1}{2}(1 - pr\cos\theta)\mathcal{H}(\lambda_-) \tag{35}$$

where $\mathcal{H}(x) = \log 2 - \frac{1}{2}(1+x)\log(1+x) - \frac{1}{2}(1-x)\log(1-x)$ and the parameters $\lambda_+$ and $\lambda_-$ are given by $\lambda_\pm = \sqrt{1 - (1-p^2)(1-r^2)/(1 \pm pr\cos\theta)^2}$. It is easy to see from equations (33)–(35) that the Shannon mutual information and the coherent information satisfy the following relations:

$$p = 0 \Rightarrow I(X^S_{\rm in} : Y^A_{\rm out}) = 0 \qquad I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow}) = I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\downarrow}) = \langle I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow\downarrow})\rangle_{\uparrow\downarrow} = S(\hat{\rho}^S_{\rm in}) \tag{36}$$

$$p = 1 \Rightarrow I(X^S_{\rm in} : Y^A_{\rm out}) = H(X^S_{\rm in}) \qquad I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow}) = I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\downarrow}) = \langle I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow\downarrow})\rangle_{\uparrow\downarrow} = 0. \tag{37}$$

In figure 1, the several quantities in equations (33)–(35) are plotted as a function of the parameter $p$ which appeared in the conditional probability. The figure clearly shows that the coherent information of the noisy quantum channel decreases as the information gain by the quantum measurement increases. In this case, the numerical calculation shows that the inequality $S(\hat{\rho}^S_{\rm in}) \leqslant I(X^S_{\rm in} : Y^A_{\rm out}) + \langle I_C(\hat{\rho}^S_{\rm in}, \hat{\mathcal{L}}^S_{\uparrow\downarrow})\rangle_{\uparrow\downarrow} \leqslant H(X^S_{\rm in})$ is established.

## References


[1] Davies E B 1976 *Quantum Theory of Open Systems* (New York: Academic)
[2] Kraus K 1983 *States, Effects, and Operations* (Berlin: Springer)
[3] Ozawa M 1984 *J. Math. Phys.* **25** 79
[4] Schumacher B 1996 *Phys. Rev.* A **54** 2614
[5] Schumacher B and Nielsen M A 1996 *Phys. Rev.* A **54** 2629
[6] Lloyd S 1997 *Phys. Rev.* A **55** 1613
[7] Barnum H, Nielsen M A and Schumacher B 1998 *Phys. Rev.* A **57** 4153
[8] Adami C and Cerf N J 1997 *Phys. Rev.* A **56** 3470
[9] Nielsen M A and Caves C M 1997 *Phys. Rev.* A **55** 2547
[10] Nielsen M A, Caves C M, Schumacher B and Barnum H 1998 *Proc. R. Soc.* A **454** 277
[11] Ban M 1998 *Int. J. Theor. Phys.* **37** 2491
[12] Ban M 1999 *J. Phys. A: Math. Gen.* **32** 1643
[13] Ban M 1999 *J. Phys. A: Math. Gen.* **32** L143
[14] Knill E and Laflamme R 1997 *Phys. Rev.* A **55** 900